Tutorial 11.

Preliminary:

① Term Structure.

$S_0(t)$ is the spot rate for a t-year maturity zero coupon bond.

The set $\{S_0(t)\}_{t>0}$ is the term structure of interest rate.

② Relationship between spot rates and yield to maturity

If the face amount of the bond is F, coupon rate is v, yield to maturity y_v has

$$P = Fv[(1+y_v)^{-1} + \cdots + (1+y_v)^{-k}] + (F+Fv)(1+y_v)^{-(k+1)}$$

$$= Fv[(1+S_0(1))^{-1} + (1+S_0(2))^{-2} + \cdots + (1+S_0(k))^{-k}] + (F+Fv)(1+S_0(k+1))^{-(k+1)}.$$

③ Forward rate

$$1 + i_0(n-1,n) = \frac{(1+S_0(n))^n}{(1+S_0(n-1))^{n-1}}.$$

Exercise:

6.1.4.

(a) $$P = Fv \cdot \sum_{k=1}^{n} (1+S_0(k))^{-k} + F(1+S_0(n))^{-n}.$$

Bond 1: $v = \frac{4\%}{2} = 2\%$, $P = 85.12$. then

$$85.12 = 0.02 \times 100 \sum_{k=1}^{20} (1+S_0(k))^{-k} + 100(1+S_0(20))^{-20}$$

Bond 2: $v = \frac{10\%}{2} = 5\%$, $P = 133.34$. then

$$133.34 = 0.05 \times 100 \sum_{k=1}^{20} (1+S_0(k))^{-k} + 100(1+S_0(20))^{-20}$$

$$\Rightarrow S_0(20) = 0.0323.$$

Yield rate 10-year zero coupon bond is $2 \times S_0(20) = 0.0646$.

(b) Assume $F=1$, then

$$P = r\sum_{k=1}^{n}(1+S_0(k))^{-k} + (1+S_0(n))^{-n}.$$

For yield j,

$$P = (1+j)^{-n} + r\sum_{k=1}^{n}(1+j)^{-k}, \text{ then}$$

Set a function $f(x) = (1+x)^{-n} + r\sum_{k=1}^{n}(1+x)^{-k}$ which is monotonically decreasing with x.

Since $S_0(1) = S_0(2) = \cdots = S_0(n-1) < S_0(n)$, then $P = r\sum_{k=1}^{n}(1+S_0(1))^{-k} + (1+S_0(n))^{-n} < r\sum_{k=1}^{n}(1+S_0(n-1))^{-k} + (1+S_0(1))^{-}$

$> r\sum_{k=1}^{n}(1+S_0(n))^{-k} + (1+S_0(n-1)$

$f(S_0(n-1)) > P = f(j) > f(S_0(n))$. $\Rightarrow$ $S_0(n-1) < j < S_0(n)$.

6.1.5.

For 1/2 year : $S_0(1) = 0.05$

For 1 year : $P = \frac{6\%/2}{1+\frac{0.10}{2}} + \frac{1+6\%/2}{(1+\frac{0.10}{2})^2} = \frac{6\%/2}{1+\frac{S_0(1)}{2}} + \frac{6\%/2}{(1+\frac{S_0(2)}{2})^2}$

$\Rightarrow S_0(1) = 0.1008$

For 1½ year : $P = \frac{4\%/2}{1+\frac{0.15}{2}} + \frac{4\%/2}{(1+\frac{0.15}{2})^2} + \frac{4\%/2+1}{(1+\frac{0.15}{2})^3} = \frac{4\%/2}{1+\frac{S_0(1)}{2}} + \frac{4\%/2}{(1+\frac{S_0(2)}{2})^2} + \frac{4\%/2+1}{(1+\frac{S_0(3)}{2})^3}$

$\Rightarrow S_0(3) = 0.15151$.

For 2 year : $P = \frac{8\%/2}{1+\frac{0.15}{2}} + \cdots + \frac{8\%/2+1}{(1+\frac{0.15}{2})^4} = \frac{8\%/2}{(1+\frac{S_0(1)}{2})} + \frac{8\%/2}{(1+\frac{S_0(2)}{2})^2} + \frac{8\%/2}{(1+\frac{S_0(3)}{2})^3} + \frac{8\%/2}{(1+\frac{S_0(4)}{2})^4}$

$\Rightarrow S_0(4) = 0.15234$.

Hence, the term structure see $\{S_0(1), S_0(2), S_0(3), S_0(4)\} = \{0.05, 0.1008, 0.15151, 0.15234\}$.

6.4.4.

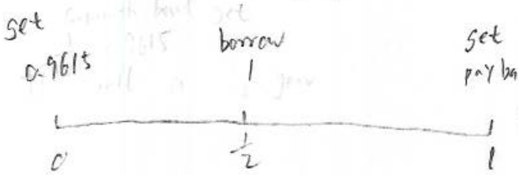
(a) $S_0(1) = \frac{8\%}{2} = 0.04$, $S_0(2) = \frac{10\%}{2} = 0.05$, $S_0(3) = \frac{x\%}{2}$. $y_r = \frac{11\%}{2} = 0.055$, $r = \frac{10\%}{2} = 0.05$.

$$P = 1.05\times(1+S_0(3))^{-3} + 0.05\times[(1+S_0(1))^{-1} + (1+S_0(2))^{-2}] = 1.05\times(1+y_r)^{-3} + 0.05\times[(1+y_r)^{-1} + (1+y_r)^{-2}]$$

$\Rightarrow x\% = 11.09\%$.

(b) $i_0(2,3) = \frac{(1+S_0(3))^3}{(1+S_0(2))^2} - 1 = \frac{(1+\frac{x\%}{2})^3}{(1+0.05)^2} - 1 = 0.11 \Rightarrow x\% = 10.33\%$.

(c)



0 : sell a 6-month bond get $\frac{1}{1.04} = 0.9615$.
buy a 1-year bond

$\frac{1}{2}$: borrow 1 with interest rate 10%.

1 : get $0.9615\times(1+\frac{10\%}{2})^2 = 1.06$ from bond, pay back $1+\frac{0.1}{2} = 1.05$.

Problem Set 10: 6.1.1, 6.1.2, 6.1.3, 6.3.1, 6.3.2. 6.3.3, 6.3.4, 6.3.5, 6.3.6, 6.4.1, 6.4.2, 6.4.3.

Tutorial: 6.1.4, 6.1.5, 6.4.4.

6.1.1

$t_i = 1$. $$C_1 (1+S_0(t_1))^{-t_1} + C_2 (1+S_0(t_2))^{-t_2} + \cdots + C_n (1+S_0(t_n))^{-t_n} = P.$$

$$\frac{100\times 10\%}{(1+S_0(1))^1} + \frac{100\times 10\%}{(1+S_0(2))^2} + \frac{100\times(1+10\%)}{(1+S_0(3))^3} = \frac{10}{1.15} + \frac{10}{1.1^2} + \frac{110}{(1.05)^3} = 111.98$$

Since coupon is paid once per year, the yield rate j satisfies.

$111.98 = 100 v_j^3 + 10 a_{\overline{3|}j} \Rightarrow j = 0.0556$.

6.1.2.

$S_0(1) = 0.1$, $S_0(2) = 0.1$, $S_0(3) = 0.12$. $S_0(4) = 0.12$, $F = 100$, $r = 5\%$.

$$P = \frac{Fr}{1+S_0(1)} + \frac{Fr}{(1+S_0(2))^2} + \frac{Fr}{(1+S_0(3))^3} + \frac{F(1+r)}{(1+S_0(4))} = \frac{5}{1.1} + \frac{5}{1.1^2} + \frac{5}{1.12^3} + \frac{105}{1.12^4} = 78.97.$$

6.1.3.

(a) (i) $\frac{Fr}{2} = \frac{100\times 10\%}{2} = 5.$

$$\frac{5}{1+\frac{S_0(0.5)}{2}} + \frac{5}{(1+\frac{S_0(1)}{2})^2} + \cdots + \frac{105}{(1+\frac{S_0(3)}{2})^6} = 5\left(\frac{1}{1.0375} + \cdots + \frac{1}{1.04125^5}\right) + \frac{105}{1.0415^6} = 104.05$$

(ii) $\frac{5}{1.07} + \frac{5}{1.0675^2} + \cdots + \frac{105}{1.06375^6} = 93.15.$

(iii) $5\cdot a_{\overline{6|}0.06} + 100(1.06)^{-6} = 95.08.$

(b) (i) $5 a_{\overline{6|}j} + 100 v_j^6 = 104.05 \Rightarrow j^{(2)} = 2j = 8.44\%.$

(ii) $5 a_{\overline{6|}j} + 100 v_j^6 = 93.15 \Rightarrow j^{(2)} = 2j = 12.82\%.$

(iii) $j = S_0(0.5) = 12\%.$

6.3.1.

Forward rate of interest: $1 + i_0(n-1, n) = \frac{(1+S_0(n))^n}{(1+S_0(n-1))^{n-1}}$.

(a) $i_0(k-1, k) = \frac{(1+S_0(k))^k}{(1+S_0(k-1))^{k-1}} - 1.$

(b) $1 + i_0(0,1) = \frac{(1+S_0(1))^1}{(1+S_0(0))^0}$, $1 + i_0(1,2) = \frac{(1+S_2(2))^2}{(1+S_0(1))^1}$, $\cdots$, $1 + i_0(k-1,k) = \frac{(1+S_k(k))^k}{(1+S_{k-1}(k-1))^{k-1}}$,

$(1+i_0(0,1)) \cdots (1+i_0(k-1,k)) = (1+S_0(k))^k.$

(c) $\frac{d}{dS_0(k)} i_0(k-1,k) = \frac{d}{dS_0(k)} \frac{(1+S_0(k))^k}{(1+S_0(k-1))^{k-1}} = \frac{k(1+S_0(k))^{k-1}}{(1+S_0(k-1))^{k-1}} > 0$, and

$\frac{d}{dS_0(k-1)} i_0(k-1,k) = \frac{d}{dS_0(k-1)} \frac{(1+S_0(k))^k}{(1+S_0(k-1))^{k-1}} = \frac{-(k-1)(1+S_0(k))^k}{(1+S_0(k-1))^k} < 0$.

(d) If $S_0(k) > S_0(k-1)$. then $i_0(k-1,k) = (1+S_0(k)) \left(\frac{1+S_0(k)}{1+S_0(k-1)}\right)^{k-1} - 1 > (1+S_0(k)) - 1$

$= S_0(k)$

6.3.3.

F = 100. for 6 months $S_0(1) = \frac{100}{97.8} - 1 = 0.02249$.

for 1-year $S_0(2) = \frac{100}{95.4} - 1 = 0.04822$.

$2 \times \left(\frac{(1+S_0(2))^2}{1+S_0(1)} - 1 \right) = 0.149$. 6-month forward rate.

$\wedge\ i(1,2)$.

6.3.4.

$i(1,2) = \frac{(1+0.10)^2}{1+0.08} - 1 = 0.1204$. $i(2,3) = \frac{(1+0.11)^3}{(1+0.10)^2} - 1 = 0.1303$

6.3.5.

(a) (i) $i_0(1,2) = \frac{(1+7\%)^2}{1+6\%} - 1 = 0.0801$

(ii) $i_0(2,3) = \frac{(1+9\%)^3}{(1+7\%)^2} - 1 = 0.1311$

(b) $i_0(3,4) = \frac{(1+S_0(4))^4}{1.09^3} - 1 \geq 0.1311 \Rightarrow S_0(4) \geq 0.1001$

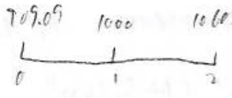
6.3.6.

$P = 5\left(\frac{1}{1.1} + \frac{1}{1.1^2} + \frac{1}{1.12^3} + \frac{1}{1.12^4}\right) + 105 \cdot \frac{1}{(1+S_0(5))^5}$. since $P = 73.68$. then

$S_0(5) = 0.125$. $i_0(4,5) = \frac{(1+0.125)^5}{(1+0.12)^4} - 1 = 0.1452$.

6.4.1.

From (i) we receive $\frac{1000}{1+10\%} = 909.09$. after (ii) $909.09(1.08)^2 = 1060.36$.



$\frac{1060.36}{1000} - 1 = 6.04\%$

(a).

6.4.2.

$S_0(2) = 7\%$. $S_0(1) = 6\%$. Since $i(1,2) = \frac{(1+S_0(2))^2}{1+S_0(1)} - 1 = 0.08$. ① Sell one-year zero coupon bond get $\frac{1}{1+6\%} = 0.9434$.

② Invest 0.9434 to two-year bond $0.9434 \times (1.07)^2 = 1.0801$. ③ borrow 1 with "someone" to pay the sold coupon bond.

④ pay the loan 1.07. get $1.08 - 1.07 = 0.01$ without any initial investment.

6.4.3.

(i) $\frac{1000}{(1.10)^2} = 826.45$ (ii) $826.45 \times (1+8\%) = 892.56$ $i(1,2) = \frac{1000}{892.56} - 1 = 12\%$. (b)